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Well-posedness of Cauchy problems for Korteweg-de Vries-Benjamin-Ono equation and Hirota equation

HUO Zhaohui[1] and GUO Boling[2 *]

(1. Graduate School, China Academy of Engineering Physics, P. O. Box 2101, Beijing 100088, China; 2. Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China)

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Abstract The well-posedness of the Cauchy problems to the Korteweg-de Vries-Benjamin-Ono equation and Hirota equation is considered. For the Korteweg-de Vries-Benjamin-Ono equation, local result is established for data in $H^s(R)\left(s\geqslant -\frac{1}{8}\right)$. Moreover, the global well-posedness for data in $L^2(R)$ can be obtained. For Hirota equation, local result is established for initial data in $H^s\left(s\geqslant\frac{1}{4}\right)$. In addition, the local solution is proved to be global in $H^s(s\geqslant 1)$ if the initial data are in $H^s(s\geqslant 1)$ by energy inequality and the generalization of the trilinear estimates associated with the Fourier restriction norm method.

Keywords: **Korteweg-de Vries-Benjamin-Ono equation, Hirota equation, the Fourier restriction norm, low regularity data.**

1 Introduction

We investigate the existence and uniqueness of the solutions to the Cauchy problems of the following nonlinear dispersive equations

$$\partial_t u + \alpha \mathscr{H}(\partial_x^2 u) + \beta \partial_x^3 u + \partial_x(u^2) = 0, \quad (1)$$

$$\partial_t u + \mathrm{i}\alpha \partial_x^2 u + \beta \partial_x^3 u + \gamma \partial_x(|u|^2 u) = 0, \quad (2)$$

respectively with the initial value

$$u(x,0) = \varphi(x) \in H^s, \quad (3)$$

where $x\in R$, $t\in R$. β, α, γ are real constants. $P.V.$ denotes the Cauchy principal value, $\mathscr{H}$ denotes the Hilbert transform

$$\mathscr{H}f(x) = P.V.\frac{1}{\pi}\int\frac{f(x-y)}{y}\mathrm{d}y.$$

Korteweg-de Vries-Benjamin-Ono equation models the undirectional propagation of long waves in a two-fluid system, where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin in the study on gravity-capillary surface waves of solitary type on deep water. Several efforts are devoted to study of existence, stability and asymptotic of solitary waves solutions of (1) ~ (3), see for instance in Refs. [1, 2]. Linares[3] showed that there exist unique local and global solutions for the Cauchy problem of (1) ~ (3) for initial data in L^2 with constant coefficients $\alpha\cdot\beta > 0$. In this paper, we are only interested in the study of well-posedness for the Cauchy problem with low regularity data here. We will prove that the Cauchy problem of (1) ~ (3) is locally well-posed in $H^s\left(-\frac{1}{8}\leqslant s\right)$ and globally well-posed in L^2 without the condition $\alpha\cdot\beta > 0$ by using the Fourier restriction norm method and the contraction mapping principle. The Fourier restriction norm method was first introduced by Bourgain[4] to study the KdV and nonlinear Schrödinger equations in the periodic case. It was simplified by Kenig, Ponce and Vega in Refs. [5, 6]. We follow their ideas to prove our results.

Hirota equation is a typical model of mathematical physics, which encompasses the well-known nonlinear Schrödinger equation and the modified KdV equation, and especially contains the nonlinear derivative Schrödinger equation[7~9]. Lots of work concerns the local and global existence and uniqueness of solutions of the equations. In Ref. [10], Guo and Tan obtained the global smooth solution of the Cauchy problem of (2) ~ (3) by the energy method. In Ref. [11], Laurey obtained the local well-posedness in $H^s\left(s > \frac{3}{4}\right)$ and global well-posedness in

* To whom correspondence should be addressed. E-mail: gbl@mail.iapcm.ac.cn

$H^s(s\geqslant 2, s=1)$ for the Cauchy problem of (2) ~ (3). Therefore, we only consider global well-posedness of Cauchy problem of (2) ~ (3) in $H^s(1\leqslant s\leqslant 2)$ here.

In this paper, we will prove the local well-posedness of Cauchy problem of (2) ~ (3) in $H^s\left(s\geqslant\frac{1}{4}\right)$ by the Fourier restriction norm method. Moreover if the initial data $\varphi(x)\in H^1$, then we can prove that there exists a unique local solution to the Cauchy problem of (2) ~ (3) and its a priori estimates. Therefore, we can extend the local solution to be global in H^1. For initial data $\varphi(x)\in H^s(1< s\leqslant 2)\subset H^1$, first we can obtain that there exists a unique local solution $u\in C((0, T_0); H^1)$, and the solution satisfies $\|u\|_{H^1}\leqslant C$ by the energy inequality. Using the properties of the solution and the generalization of trilinear estimates, we can prove that the above solution belongs to $C((0, T_0); H^s)$ $(1< s\leqslant 2)$. In order to extend the solution of (2) ~ (3) to any time $T>0$ in H^s, we make the iteration scheme as follows: take $u(T_0)$ as the initial data, and get the local solution $u\in C((T_0, T_1); H^1)$. Similarly as above, the solution is in $C((T_0, T_1); H^s)$ $(1< s\leqslant 2)$. In fact we can continue the above process of $\frac{T}{T_0}$ steps by a priori estimate $\|u\|_{H^1}\leqslant C$.

Both of the equations have similar properties because Korteweg-de Vries-Benjamin-Ono equation and Hirota equation have the dispersive term $\partial_x^3 u$. Therefore, we can solve them by the same method.

To study the above problem, we use the integral equivalent formulations

$$u = S(t)\varphi - \int_0^t S(t-t')F(t')\mathrm{d}t',$$

where $S(t)=\mathscr{F}_x^{-1}\mathrm{e}^{\mathrm{i}t(\beta\xi^3-\alpha\xi|\xi|)}\mathscr{F}_x$ or $S(t)=\mathscr{F}_x^{-1}\mathrm{e}^{\mathrm{i}t(\alpha\xi^2+\beta\xi^3)}\mathscr{F}_x$ are the unitary operators associated to the linear equations respectively. $F(x, t)=\partial_x(u^2)$ or $\partial_x(|u|^2u)$. For simplicity, denote $\phi(\xi)=\beta\xi^3-\alpha\xi|\xi|$, or $\phi(\xi)=\alpha\xi^2+\beta\xi^3$.

We introduce the notation of Bourgain' space. For s, $b\in R$, we define the space $X_{s,b}$ as the completion of the Schwartz function space on R^2 with respect to the norm

$$\|u\|_{X_{s,b}} = \|S(-t)u\|_{H_x^sH_t^b} = \|\langle\xi\rangle^s\langle\tau-\phi(\xi)\rangle^b\mathscr{F}u\|_{L_\xi^2L_\tau^2},$$

$$\langle\cdot\rangle = (1+|\cdot|).$$

Denote $\hat{u}(\tau,\xi)=\mathscr{F}u$ by the Fourier transform in t and x of u and $\mathscr{F}_{(\cdot)}u$ by the Fourier transform in the $(\cdot)$ variable. Let $\psi\in C_0^\infty(R)$ with $\psi=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ and supp $\psi\subset[-1, 1]$. $\psi_\delta(t)=\psi\left(\frac{t}{\delta}\right)$.

For the Korteweg-de Vries-Benjamin-Ono equation, we have the following theorems.

Theorem 1. Let $-\frac{1}{8}\leqslant s$, $\frac{1}{2}< b<\frac{5}{8}$. Then there exists a constant $T>0$, Cauchy problem of (1) ~ (3) admits a unique local solution $u(x, t)\in C((0, T); H^s)\cap X_{s,b}$ with $\varphi\in H^s$. Moreover, given $t\in(0, T)$, the mapping $\varphi\rightarrow u(t)$ is Lipschitz continuous from H^s to $C((0, T); H^s)$.

The smooth solution of Cauchy problem of (1) ~ (3) is proved to satisfy the L^2 conservation law, so is the solution to the Cauchy problem of (1) ~ (3) for $s\geqslant 0$. Then we have the global well-posedness of Cauchy problem of (1) ~ (3) for data in L^2.

Theorem 2. For $s=0$, the solution obtained in Theorem 1 can be extended for any $T>0$.

For the Hirota equation, we have

Theorem 3. Let $s\geqslant\frac{1}{4}$, $\frac{1}{2}< b<\frac{5}{8}$. Then there exists a constant $T>0$, such that the Cauchy problem of (2) ~ (3) admits a unique local solution $u(x, t)\in C((0, T); H^s)\cap X_{s,b}$ with $\varphi\in H^s$. Moreover, given $t\in(0, T)$, the map $\varphi\rightarrow u(t)$ is Lipschitz continuous from H^s to $C((0, T); H^s)$.

Theorem 4. Let $s\geqslant 1$, the Cauchy problem of (2) ~ (3) is global well-posed in H^s with initial data $\varphi\in H^s(s\geqslant 1)$.

Indeed, one can obtain local well-posedness of Cauchy problems of (1) ~ (3) and (2) ~ (3) by the Picard iteration method provided that

$$\|\partial_x(u_1u_2)\|_{X_{s,b-1}}\leqslant C\|u_1\|_{X_{s,b}}\|u_2\|_{X_{s,b}}, \tag{4}$$

$$\|\partial_x(u_1u_2\bar{u}_3)\|_{X_{s,b-1}} \leqslant C\|u_1\|_{X_{s,b}}\|u_2\|_{X_{s,b}}\|u_3\|_{X_{s,b}}, \tag{5}$$

hold for some $b>\frac{1}{2}$. We only need to prove the multi-linear estimates (4) and (5) to obtain the local well-posedness of Cauchy problems. Therefore, we need the following preliminary estimates.

2 Preliminary estimates

In this section, we shall deduce several estimates. Let us use the notations

$$P^N f=\int_{|\xi|\geqslant N} e^{ix\xi}\hat{f}(\xi)d\xi,$$

$$P_N f=\int_{|\xi|\leqslant N} e^{ix\xi}\hat{f}(\xi)d\xi,$$

$$\|f\|_{L_x^pL_t^q}=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x,t)|^q dt\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}},$$

$$\|f\|_{L_t^\infty H_x^s}=\|\,\|f\|_{H_x^s}\,\|_{L_t^\infty},$$

$$\mathscr{F}F_\rho(\xi,\tau)=\frac{f(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}$$

a is a constant, which depends on α, β.

Lemma 1. [12] The group $\{S(t)\}_{-\infty}^{+\infty}$ satisfies

$$\|S(t)\varphi\|_{L_x^8L_t^8}\leqslant\|\varphi\|_{L^2}. \tag{6}$$

Lemma 2.

$$\|D_xS(t)P^{2a}\varphi\|_{L_x^\infty L_t^2}\leqslant\|\varphi\|_{L^2}, \tag{7}$$

$$\|D_x^{-\frac{1}{4}}S(t)P^a\varphi\|_{L_x^4L_t^\infty}\leqslant\|\varphi\|_{L^2}, \tag{8}$$

$$\|D_x^{\frac{1}{6}}P^{2a}S(t)\varphi\|_{L_x^6L_t^6}\leqslant\|\varphi\|_{L^2}. \tag{9}$$

Remark. In the proof of Lemma 2, we introduce the operator P^N to eliminate the nonzero singular point of phase function $\phi(\xi)$. We mainly refer to Ref.[13].

Lemma 3. If $\rho>\frac{1}{2}$, $\forall N>0$

$$\|P_NF_\rho\|_{L_x^2L_t^\infty}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

Lemma 4. (i) If $\rho>\frac{1}{3}$, then

$$\|F_\rho\|_{L_x^4L_t^4}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

(ii) If $\rho>\frac{2}{9}$, then

$$\|F_\rho\|_{L_x^3L_t^3}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

Lemma 5. (i) Let $\rho>\frac{\theta}{2}$ with $\theta\in[0,1]$. Then

$$\|D_x^\theta P^{2a}F_\rho\|_{L_x^{\frac{2}{1-\theta}}L_t^2}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

(ii) Let $\rho>\frac{1}{2}$, then

$$\|D_x^{-\frac{1}{4}}P^{2a}F_\rho\|_{L_x^4L_t^\infty}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

(iii) If $\rho>\frac{3}{8}$, then

$$\|D_x^{\frac{1}{8}}P^{2a}F_\rho\|_{L_x^4L_t^4}\leqslant C\|f\|_{L_\xi^2L_\tau^2}.$$

Remark. We can obtain Lemmas 3, 4 and 5 by Lemmas 1 and 2.

Lemma 6. We assume that functions $\bar{\hat{f}}$, $\hat{f_1}$, $\hat{f_2}$, $\hat{f_3}$ belong to schwartz space on R^2.

$$\int_{\xi=\xi_1+\xi_2+\xi_3;\,\tau=\tau_1+\tau_2+\tau_3}\bar{\hat{f}}(\xi,\tau)\hat{f_1}(\xi_1,\tau_1)\circ\hat{f_2}(\xi_2,\tau_2)\hat{f_3}(\xi_3,\tau_3)d\delta=\int ff_1f_2f_3(x,t)dxdt.$$

$$\int_{\xi=\xi_1+\xi_2;\,\tau=\tau_1+\tau_2}\bar{\hat{f}}(\xi,\tau)\hat{f_1}(\xi_1,\tau_1)\hat{f_2}(\xi_2,\tau_2)d\delta=\int\bar{f}f_1f_2(x,t)dxdt.$$

Lemma 7. Let $s\in R$, $\frac{1}{2}<b<b'<1$, $0<\delta<1$. Then we have

$$\|\psi_\delta(t)S(t)\varphi\|_{X_{s,b}}\leqslant C\delta^{\frac{1}{2}-b}\|\varphi\|_{H^s},$$

$$\left\|\psi_\delta(t)\int_0^tS(t-\tau)F(\tau)d\tau\right\|_{X_{s,b}}\leqslant C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$

$$\left\|\psi_\delta(t)\int_0^tS(t-\tau)F(\tau)d\tau\right\|_{L_t^\infty H_x^s}\leqslant C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$

$$\|\psi_\delta(t)F\|_{X_{s,b-1}}\leqslant C\delta^{b'-b}\|F\|_{X_{s,b'-1}}.$$

3 Local results

In this section, we will obtain (4) and (5), and we have the following theorems.

Theorem 5. Let $\frac{1}{2}<b$ be close enough to $\frac{1}{2}$. For $\frac{1}{2}<b'$ and $s\geqslant-\frac{1}{8}$, we have

$$\|\partial_x(u_1u_2)\|_{X_{s,b-1}}\leqslant C\|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}.$$

Theorem 6. If $s \geqslant \frac{1}{4}$, $\frac{1}{2} < b < \frac{5}{8}$, $b' > \frac{1}{2}$, then

$$\| \partial_x(u_1 u_2 \bar{u}_3) \|_{X_{s,b-1}} \leqslant C \| u_1 \|_{X_{s,b'}} \| u_2 \|_{X_{s,b'}} \| u_3 \|_{X_{s,b'}}.$$

Remark. In the proof of Theorems 5 and 6, we introduce the operator P^N, choose appropriate N, we have

$$\| D_x S(t) P^N \varphi \|_{L_x^\infty L_t^2} \leqslant C \| \varphi \|_{L^2},$$
$$\| S(t) P^N \varphi \|_{L_x^4 L_t^\infty} \leqslant C \| \varphi \|_{H^{\frac{1}{4}}},$$

which are called global and local smoothing effects of dispersive equations respectively. Therefore, we will divide the Bourgain' space into two parts ($|\xi| \leqslant N$ and $|\xi| \geqslant N$). We use the above global and local smoothing effects to prove the result in $|\xi| \geqslant N$ part (we mainly use Lemma 5 to prove it), use Strichartz estimate to prove it in $|\xi| \leqslant N$ part (we mainly use Lemmas 3 and 4 to prove it).

Next, we give the outline proofs of Theorems 1 and 3. For $\varphi \in H^s$ we define the operator

$$\Phi(u) = \psi_1(t) S(t) \varphi - \psi_1(t) \int_0^t S(t-t') \psi_\delta(t') F(t') \mathrm{d}t',$$

and the set

$$\mathscr{B} = \{ u \in X_{s,b} : \| u \|_{X_{s,b}} \leqslant C \| \varphi \|_{H^s} \}.$$

In order to show that Φ is a contraction mapping on $\mathscr{B}$ we first prove

$$\Phi(\mathscr{B}) \subset \mathscr{B}.$$

We first consider linear estimates (Lemma 7)

$$\| \psi_\delta(t) S(t) \varphi \|_{X_{s,b}} \leqslant C \delta^{\frac{1}{2}-b} \| \varphi \|_{H^s},$$

$$\left\| \psi_\delta(t) \int_0^t S(t-t') F(t') \mathrm{d}t' \right\|_{X_{s,b}} \leqslant C \delta^{\frac{1}{2}-b} \| F \|_{X_{s,b-1}},$$

$$\left\| \psi_\delta(t) \int_0^t S(t-t') F(t') \mathrm{d}t' \right\|_{L_t^\infty H_x^s} \leqslant C \delta^{\frac{1}{2}-b} \| F \|_{X_{s,b-1}}.$$

Next, we consider nonlinear estimates (Theorems 5 and 6).

For the Korteweg-de Vries-Benjamin-Ono equation, we have

$$\| \Phi(u) \|_{X_{s,b}} \leqslant C \| \varphi \|_{H^s} + C \delta^{b'-b} \| u \|_{X_{s,b}}^2 \leqslant C \| \varphi \|_{H^s} + C \delta^{b'-b} \| \varphi \|_{H^s} \| u \|_{X_{s,b}}.$$

Therefore, if fix δ such that $C \delta^{(b'-b)} \| \varphi \|_{H^s} \leqslant \frac{1}{2}$, then we have

$$\Phi(\mathscr{B}) \subset \mathscr{B}.$$

Let $u, v \in \mathscr{B}$ in an analogous way to above, we obtain

$$\| \Phi(u) - \Phi(v) \|_{X_{s,b}} \leqslant \frac{1}{2} \| u - v \|_{X_{s,b}}.$$

Therefore, Φ is a contraction mapping on $\mathscr{B}$. There exists a unique fixed point which solves the Cauchy problem for $T < \delta$.

For the Hirota equation, we have

$$\| \Phi(u) \|_{X_{s,b}} \leqslant C \| \varphi \|_{H^s} + C \delta^{b'-b} \| u \|_{X_{s,b}}^3 \leqslant C \| \varphi \|_{H^s} + C \delta^{b'-b} \| \varphi \|_{H^s}^2 \| u \|_{X_{s,b}},$$

therefore, if we fix δ such that $C \delta^{b'-b} \| \varphi \|_{H^s}^2 \leqslant \frac{1}{2}$, then

$$\Phi(\mathscr{B}) \subset \mathscr{B}.$$

Similarly with the Korteweg-de Vries-Benjamin-Ono equation, Φ is a contraction mapping on $\mathscr{B}$. There exists a unique fixed point which solves the Cauchy problem for $T < \delta$.

4 Global solution in H^s ($1 \leqslant s \leqslant 2$) of the Hirota equation

In this section, we have the following generalization of the trilinear estimate of the Hirota equation.

Lemma 8. Let $s \geqslant 0$, $\frac{1}{2} < b \leqslant \frac{2}{3}$, $b' > \frac{1}{2}$. Then we have

$$\| u_1 u_2 \partial_x(\bar{u}_3) \|_{X_{s,b-1}} \leqslant C \| u_1 \|_{X_{s_1,b'}} \| u_2 \|_{X_{s_2,b'}} \| u_3 \|_{X_{s_3,b'}}, \tag{10}$$

$$\| u_1 \bar{u}_2 \partial_x(u_3) \|_{X_{s,b-1}} \leqslant C \| u_1 \|_{X_{s_1,b'}} \| u_2 \|_{X_{s_2,b'}} \| u_3 \|_{X_{s_3,b'}}, \tag{11}$$

where we choose non-negative different numbers (s_1, s_2, s_3) to satisfy the following cases:

$$s_1 + s_3 \geqslant 2b - 1, \quad s_2 + s_3 \geqslant 2b - 1, \tag{12}$$
$$s - s_2 \leqslant 1, \quad s - s_1 \leqslant 1, \tag{13}$$
$$1 + s_1 \geqslant s, \quad 1 + s_2 \geqslant s, \tag{14}$$

$$s_1 + s_3 \geqslant \frac{2}{3}, \quad s_2 + s_3 \geqslant \frac{2}{3}, \tag{15}$$

$$s = s_3, \tag{16}$$

$$s - \min\{1 - b, s_2\} \leqslant s_1 + s_3 + 1 - 2b, \tag{17}$$

$$1 - \min\{1 - b, s_2\} - s_3 \leqslant s_1 - s + 2(1 - b). \tag{18}$$

In fact, we choose $1 \leqslant s = s_3 \leqslant 2$, $1 \leqslant s_1$, $s_2 \leqslant 2$, then (10) and (11) hold.

Lemma 9. If initial data $\varphi \in H^1$, then Cauchy problem of (2) ~ (3) is local well-posed. Moreover, the solution u satisfies $\| u \|_{H^1} \leqslant C$.

Remark. We use generalization of the trilinear estimate to make the local solution u of the Cauchy problem of (2) ~ (3) global in H^s $(1 < s \leqslant 2)$ for data in H^s by energy inequality of solution u in H^1. Therefore, we can obtain the solution $u \in C((0, \infty); H^s)$ for initial data $\varphi \in H^s$ $(1 \leqslant s \leqslant 2)$.

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